half of the plate cross section, remains finite. Thus $\bar{Q}_\theta'$ has a built-in factor a/h from normalization. Since the right-hand side of (A5) and $\bar{Q}_\theta'$ ($\approx 3aQ/2hF$) approach infinity in the same order of a/h, the actual shear Q_θ remains finite.

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Constants of the Motion for Optimum Thrust Trajectories in a Central Force Field

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This paper derives four constants of the motion for optimal thrust trajectories in a central force field. Two additional constants of the motion are derived which hold for singular thrusting arcs as well as impulsive thrusts. The paper applies the constants of the motion for the impulsive thrust case to obtain a set of initial conditions for the classical adjoint variables to be used as a good approximation for a solution of the finite thrust arc by the indirect method.

Introduction

THE constants of the motion of a system of differential equations play an important role in characterizing the solutions. This paper develops an application of the constants of the motion to the indirect methods for obtaining solutions of optimal thrust trajectories by iterative procedures.

The optimal trajectories for a thrusting vehicle in a central force field have been under study for some time by Lawden,[1, 2] Leitmann,[3,4] Melbourne,[5] Breakwell,[6] and others.† Four constants of the motion for this problem are well known. This paper derives two additional constants of the motion which hold for singular thrusting arcs and impulsive thrusts. The paper also derives the four known constants of the motion. The paper applies the constants of the motion for the impulsive thrust case to obtain a set of initial conditions for the classical adjoint variables to be used as a good approximation for the solution of the finite thrust arc by the indirect method.

As is well known, the indirect methods for obtaining solutions of the optimal thrust trajectories by iterative procedures suffer from an extreme sensitivity of the solution to small changes in the initial conditions of the adjoint variables. In effect, the success of the gradient techniques, employed by Kelley[7] and Bryson,[8] is largely due to their ability to control the incremental step size for small changes in the thrusting logic.

Once a good approximation to the optimum control thrust logic has been obtained, the gradient techniques prove too slow for convergence and resort is made to the classical indirect methods for the last few iterations. If a good approximation to the initial conditions of the adjoint variables were available, the indirect methods would be in more general use.

A good approximation to the initial conditions of the adjoint variables for a given problem may be obtained through a study of the limiting impulsive solution to the same problem. Let us assume that a solution to an optimal thrust trajectory exists and that it is known. Then, if one could improve the efficiency of the engine (thrust/weight ratio), a shorter burning arc could be obtained for an improved optimal trajectory. In the limit one would obtain the impulsive thrust solution of the given problem which would indeed require a perfect engine. Thus, we can look at the impulsive solution as a limiting point in a simply connected region in the space of the initial conditions of the adjoint variables. Intuitively, one might expect that an iterative procedure could be developed which would start with the known impulsive solution and converge to the required finite thrust solution.

This report applies the constants of the motion for an optimal impulsive trajectory to obtain approximate values

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† The author has been informed of a similar study by Hillsley and Robbins of IBM applying impulsive thrusts to optimum fuel trajectories for bounded thrust and fixed transfer time.

of the adjoint variables to be used for an indirect method solution of the optimal trajectory with finite thrust. The specific example illustrated in this report is the minimum fuel required for bounded thrust between fixed initial and final position and velocity states with time open, but bounded. It is necessary to require a finite transfer time in order to exclude the "oddball" solution consisting of an infinite number of infinitesimal impulses extending over an infinite transfer time.

I. Equations of Motion

The equations of motion of a thrusting vehicle in a central force field are given by

$$\ddot{R} = -\mu(R/r^3) + (k/m)T \qquad \dot{m} = -k/c \tag{1}$$

where $|T| = 1$, and $c = \text{const}$.

The necessary conditions for minimizing the fuel consumption for bounded thrust with time fixed, or minimum time for fixed fuel, are given by

$$T = \lambda/|\lambda| \tag{2}$$

and

$$\begin{aligned} k &= k_{\max} \quad \text{if} \quad [|\lambda| - (m\sigma/c) > 0] \\ k &= k_{\min} \quad \text{if} \quad [|\lambda| - (m\sigma/c) < 0] \\ k_{\min} \leq k &\leq k_{\max} \quad \text{if} \quad (|\lambda| \equiv m\sigma/c) \end{aligned} \tag{3}$$

The adjoint variables are solutions of the Euler-Lagrange differential equations as follows:

$$\ddot{\lambda} = -\mu\frac{\lambda}{r^3} + 3\mu\frac{\lambda \cdot R}{r^5}R \qquad \dot{\sigma} = \frac{k}{m^2}\lambda\cdot T \tag{4}$$

The final equations that yield the optimum trajectories (if any exist at all) are given by

$$\ddot{R} = -\mu\frac{R}{r^3} + \frac{k}{m}\frac{\lambda}{|\lambda|} \tag{5a}$$

$$\ddot{\lambda} = -\mu\frac{\lambda}{r^3} + 3\mu\frac{R\cdot\lambda}{r^5}R \tag{5b}$$

$$\dot{m} = -k/c \tag{5c}$$

$$\dot{\sigma} = k\,|\lambda|/m^2 \tag{5d}$$

To obtain the proper solution, it is necessary to make some statement about the initial and final conditions of the state variables. For the purposes of this paper, it will be sufficient to characterize all the solutions of the equations of motion through the constants of the motion. For this reason, little discussion of the initial, final, or transversality conditions will be carried out.

II. The Constants of the Motion

This section contains a derivation of four constants of the motion of Eqs. (5a–5d). For the special case of the singular thrusting arc and the impulsive solutions, two more constants of the motion are given.

By forming the vector cross product of λ with Eq. (5a), the vector cross product of R with Eq. (5b), and adding, the following equation results:

$$\lambda + \ddot{R} + R \times \ddot{\lambda} = -\mu\frac{\lambda \times R}{r^3} - \mu\frac{R \times \lambda}{r^3} = 0 \tag{6}$$

Thus, three constants of the motion are given by the vector equation

$$(d/dt)(\lambda \times \dot{R} + R \times \dot{\lambda}) = 0 \tag{7}$$

The equation may be written as a vector constant

$$\lambda \times \dot{R} + R \times \dot{\lambda} = A \tag{8}$$

In order to obtain a more convenient form for the optimal thrust logic, Eqs. (5c) and (5d) may be combined as follows:

$$\begin{aligned} (d/dt)(m\sigma) &= \dot{m}\sigma + m\dot{\sigma} \\ &= -k(\sigma/c) + k(|\lambda|/m) \end{aligned} \tag{9}$$

Thus,

$$(d/dt)(m\sigma) = (k/m)[|\lambda| - (m\sigma/c)] \tag{10}$$

If the coefficient of k is positive, we use $k_{\max}$, if the coefficient is negative, we use $k_{\min}$. In either case, k is a constant so long as its coefficient is not identically zero. On the other hand, if the coefficient of k is identically zero, then

$$(d/dt)(m\sigma) \equiv 0 \tag{11}$$

This condition is satisfied along a singular arc so that

$$m\sigma = \text{const} \tag{12}$$

Since $|\lambda| - (m\sigma/c) \equiv 0$, it follows that $|\lambda|$ is a constant. Thus the optimum thrust logic may be stated simply as follows: either

$$k = \text{const} \quad \text{if} \quad |\lambda| - (m\sigma/c) \neq 0 \tag{13a}$$

or

$$|\lambda| = \text{const} \quad \text{if} \quad |\lambda| - (m\sigma/c) \equiv 0 \tag{13b}$$

It is now possible to obtain the fourth constant of the motion. Form the dot product of Eq. (5a) with $\dot{\lambda}$, the dot product of Eq. (5b) with $\dot{R}$, and add. The result is

$$\begin{aligned} \dot{\lambda}\cdot\ddot{R} + \dot{R}\cdot\ddot{\lambda} &= \frac{d}{dt}(\dot{\lambda}\cdot\dot{R}) \\ &= -\mu\frac{d}{dt}\left(\frac{\lambda\cdot R}{r^3}\right) + \frac{k}{m}\frac{d}{dt}|\lambda| \end{aligned} \tag{14}$$

Since

$$\begin{aligned} k\cdot\frac{d}{dt}\frac{|\lambda|}{m} &= \frac{k}{m}\frac{d}{dt}|\lambda| + k|\lambda|\frac{d}{dt}\left(\frac{1}{m}\right) \\ k\frac{d}{dt}\frac{\sigma}{c} &= k|\lambda|\frac{d}{dt}\left(\frac{1}{m}\right) \end{aligned} \tag{15}$$

it follows that

$$\begin{aligned} k\frac{d}{dt}\left(\frac{|\lambda|}{m} - \frac{\sigma}{c}\right) &= \frac{k}{m}\frac{d}{dt}|\lambda| \\ &= \frac{d^2}{dt^2}m\sigma \end{aligned} \tag{16}$$

Thus, a fourth constant of the motion is given by

$$\dot{\lambda}\cdot\dot{R} + \mu\frac{R\cdot\lambda}{r^3} - \frac{d}{dt}m\sigma = h \tag{17}$$

From Eq. (10), an altered form of this constant of the motion is

$$\dot{\lambda}\cdot\dot{R} + \mu\frac{R\cdot\lambda}{r^3} - k\left(\frac{|\lambda|}{m} - \frac{\sigma}{c}\right) = h \tag{17a}$$

This is the so-called Hamiltonian. In particular, for the singular arc

$$\begin{aligned} d/dt(m\sigma) &= 0 \\ \dot{\lambda}\cdot\dot{R} + \mu[(R\cdot\lambda)/r^3] &= h \end{aligned} \tag{18}$$

Another constant of the motion may be obtained for some restricted cases. Form the dot product of Eq. (5a) with

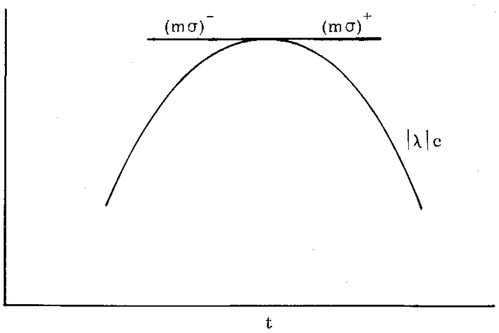


Fig. 1 Interior impulse.

λ, the dot product of Eq. (5b) with R, and subtract. The result is

$$\lambda\cdot\ddot{R} - R\cdot\ddot{\lambda} = \frac{d}{dt}(\lambda\cdot\dot{R} - R\cdot\dot{\lambda}) = -3\mu\frac{\lambda\cdot R}{r^3} + \frac{k}{m}|\lambda| \tag{19}$$

In addition, we have

$$\frac{d}{dt}(\dot{\lambda}\cdot R) = \dot{\lambda}\cdot\dot{R} + \ddot{\lambda}\cdot R = \dot{\lambda}\cdot\dot{R} + 2\mu\frac{\lambda\cdot R}{r^3} \tag{20}$$

It is possible to eliminate $\dot{\lambda}\cdot\dot{R}$ between Eq. (18) and Eq. (20) as follows:

$$\dot{\lambda}\cdot\dot{R} = \frac{d}{dt}(\dot{\lambda}\cdot R) - 2\mu\frac{\lambda\cdot R}{r^3} = h - \mu\frac{\lambda\cdot R}{r^3} + \frac{d}{dt}(m\sigma) \tag{21}$$

It is also possible to eliminate $\mu[(\lambda\cdot R)/r^3]$ between Eq. (19) and (21) as follows:

$$\frac{1}{3}\frac{d}{dt}(\lambda\cdot\dot{R} - R\cdot\dot{\lambda}) + \frac{|\lambda|c}{3}\frac{d}{dt}(\log m) = \frac{d}{dt}(m\sigma + ht - \dot{\lambda}\cdot R) \tag{22}$$

If $|\lambda|$ is a constant (this is the case for the singular arc), we have as a fifth constant of the motion

$$\frac{1}{3}\lambda\cdot\dot{R} + \frac{2}{3}R\cdot\dot{\lambda} + \frac{|\lambda|c}{3}\log m - ht = b \tag{23}$$

Moreover, for the same restrictive case, another constant of the motion is given by

$$m\sigma = d \tag{24}$$

To obtain the form of these new constants of the motion for

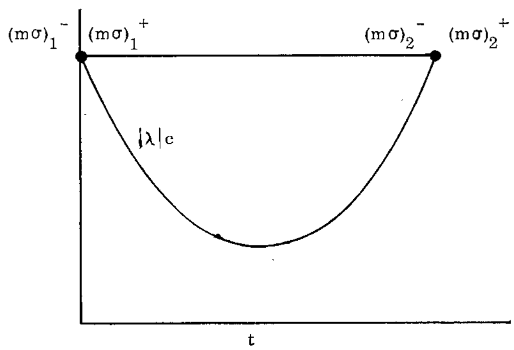


Fig. 2 Boundary impulsive thrust.

impulsive thrust, some care must be taken in approaching the limit forms.

It is necessary to distinguish between impulsive thrusts in the interior time domain between the initial and final conditions and the impulsive thrusts at the boundaries for optimum thrust trajectories executed in finite time, i.e., time open, but bounded.

Interior Impulsive Thrust

From Eq. (16) we have

$$-c\frac{d}{dt}(\log m)\frac{d}{dt}|\lambda| = \frac{d^2}{dt^2}(m\sigma) \tag{16a}$$

Integrating over an interior impulse, we have

$$-c\{\log(m^+) - \log(m^-)\}\frac{d}{dt}|\lambda| = \left(\frac{d}{dt}m\sigma\right)^+ - \frac{d}{dt}(m\sigma)^- \tag{25}$$

It is plain that during an interior coasting arc the engine is off, $k = 0$, and $(d/dt)(m\sigma) = 0$. Thus, it follows that

$$-c\{\log(m^+) - \log(m^-)\}(d/dt)|\lambda| = 0 \tag{25a}$$

Since the jump in $\log m$ is not zero, it follows that for interior impulses

$$(d/dt)|\lambda| = 0 \qquad \lambda\cdot\dot{\lambda} = 0 \tag{26}$$

Since λ is a continuous function with continuous derivatives (up through $\ddot{\lambda}$) then the maximum value of $|\lambda|$ is the same constant for the entire interior domain between the initial and final conditions (see Fig. 1).

For impulsive thrusts in the interior of the domain we have that $(d/dt)(m\sigma) = 0$. Thus, the two new constants of the motion for the impulsive case are identical to those for the singular case within the interior domain.

Boundary Impulses

For a boundary impulse, $(d/dt)(m\sigma)$ vanishes only at one end of the impulse. Thus, Eq. (25) becomes

$$-c\log\frac{m^+}{m^-}\frac{d}{dt}|\lambda| = \begin{cases}\text{either } \dfrac{d}{dt}(m\sigma)^+ \\ \text{or } -\dfrac{d}{dt}(m\sigma)^-\end{cases} \tag{25b}$$

The positive sign is associated with a terminating boundary impulse, and the negative sign is associated with an initiating boundary impulse. Since, from Eq. (10),

$$\frac{d}{dt}(m\sigma) = \frac{k}{m}\left(|\lambda| - \frac{\sigma m}{c}\right)$$

the product of an infinite, impulsive thrust and a vanishing switch function is indeterminate at an impulse. Equation (25b) may be used to evaluate this indeterminacy. At both the initial and the terminal boundary inpulses, we have

$$-c\log\frac{m^+}{m^-}\frac{d}{dt}|\lambda| = \pm\frac{k}{m}\left(|\lambda| - \frac{\sigma m}{c}\right) \tag{25c}$$

In addition, from Eq. (5d), integrating over the boundary impulse,

$$\sigma^+ - \sigma^- = c|\lambda|\left(\frac{1}{m^+} - \frac{1}{m^-}\right) \tag{27}$$

Since, at the interior boundary immediately following the impulse,

$$\sigma^+ = c|\lambda|/m^+ \tag{28}$$

it follows,

$$\sigma^- = c|\lambda|/m^- \tag{29}$$

Thus, at the boundaries, as shown in Fig. 2, we have

$$(m\sigma)^- = (m\sigma)^+ = c|\lambda| \tag{30}$$

The constants of the motion for impulsive thrusts at the boundaries are seen to be identical with those for interior thrusts as well as the singular case so long as we interpret the state variables referred to their interior values at the boundary.

The natural boundary condition for minimum fuel is given by $\sigma_f = 1$. It is now possible to obtain the natural scaling factor for $|\lambda|$ from the equation

$$|\lambda| = m_f{}^+/c \tag{31}$$

The initial value of σ may then be obtained from

$$\sigma_{\text{initial}} = m_f{}^+/m^-{}_{\text{initial}} \tag{32}$$

To summarize, the general constants of the motion (which hold for all solutions) are given by

$$\lambda \times \dot{R} + R \times \dot{\lambda} = A$$

$$\mu\,\frac{\lambda\cdot R}{r^3} + \dot{\lambda}\cdot\dot{R} - k\left(\frac{|\lambda|}{m} - \frac{\sigma}{c}\right) = h \tag{33}$$

For the special case of the singular arc and for impulsive thrusts on the interior domain, the following additional constants hold:

$$\tfrac{1}{3}\lambda\cdot\dot{R} + \tfrac{2}{3}R\cdot\dot{\lambda} + c(|\lambda|/3)\log m - ht = b$$

$$m\sigma = d \tag{34}$$

For the singular case $|\lambda|$ = const, and for the impulsive case $|\lambda|_{initial} = |\lambda|_{final}$.

III. The Impulsive Solution

Given two position vectors in space, and a central angle α, the vector velocity required to pass a free fall trajectory between the two position vectors is given by

$$\dot{R}_1 = \frac{\mu \tan\alpha/2}{p\, r_1} R_1 + \frac{p}{r_1\, r_2 \sin\alpha}(R_2 - R_1) \tag{35}$$

Conversely, the velocity vector at the other end is given by

$$\dot{R}_2 = -\frac{\mu \tan\alpha/2}{p\, r_2} R_2 + \frac{p}{r_1 r_2 \sin\alpha}(R_2 - R_1) \tag{36}$$

The value of p is the magnitude of the angular momentum,

$$p = |R \times \dot{R}| = \text{const during coast} \tag{37}$$

This parameter may be used as a variable for the purposes of differentiating the total impulse to obtain the optimal impulsive trajectory.

Given the initial vectors R_1, $\dot{R}_1{}^-$ and the final vectors R_2, $\dot{R}_2{}^+$, it is required to find the minimum fuel necessary to go from condition one to condition two in a central force field. As shown in Fig. 3, let

$$\Delta V_1 = \dot{R}_1{}^+ - \dot{R}_1^-$$

$$\Delta V_2 = \dot{R}_2{}^+ - \dot{R}_2{}^- \tag{38}$$

The scalar magnitudes of these impulsive changes in velocity are given by

$$\delta v_1 = |\Delta v_1| \qquad \delta v_2 = |\Delta V_2| \tag{39}$$

The condition for minimum fuel is

$$\frac{\partial}{\partial p}(\delta v_1 + \delta v_2) = 0 \tag{40}$$

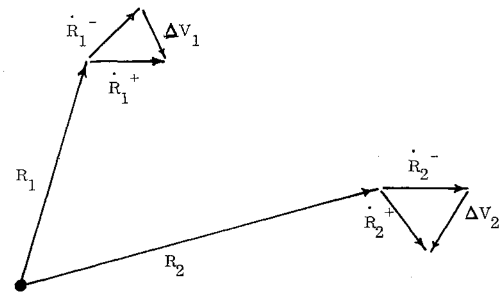


Fig. 3 Two-point transfer.

The resulting equation is given by

$$\delta v_2(\Delta V_1)\cdot\frac{\partial}{\partial p}\dot{R}_1 + \delta v_1(\Delta V_2)\cdot\frac{\partial}{\partial p}\dot{R}_2 = 0 \tag{41}$$

Equation (41) is an eighth-order polynomial in the variable p which may be solved by standard numerical techniques. For each real root, it is possible to evaluate the total scalar impulse, and we may choose the minimum of these as our solution. The change in mass required to execute each successive impulse is given by

$$m_i{}^+ = m_i{}^-\, e^{-\delta v_i/c} \tag{42}$$

IV. Initial Conditions for the Adjoint Variables for Impulsive Thrust

The impulsive change in velocity may be obtained by integrating Eq. (5a):

$$\dot{R}_1{}^+ - \dot{R}_1{}^- = -c\log(m_1{}^+/m_1{}^-)(\lambda_1/|\lambda|) \tag{43}$$

The value of $|\lambda|$ is obtained from Eq. (31):

$$|\lambda| = m_2{}^+/c$$

The initial conditions for λ are

$$\lambda_1 = \frac{m_2{}^+}{c}\,\frac{\dot{R}_1{}^+ - \dot{R}_1{}^-}{\delta v_1} \tag{44}$$

The initial value of σ is given by Eq. (32):

$$\sigma_1 = m_2{}^+/m_1{}^-$$

and is valid only for impulsive thrusts.

In order to obtain a first-order approximation to the initial value of σ for the finite thrust case, resort is made to a Taylor series expansion of σm about the initial time,

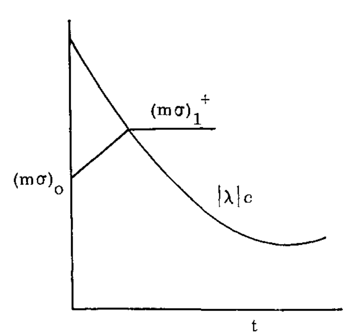
$$(\sigma m)_0{}^+ = (\sigma m)_0{}^- + (d/dt)(\sigma m)_0(t - t_0) \tag{45}$$

This condition is illustrated in Fig. 4.

From Eq. (25b),

$$\frac{d}{dt}(\sigma m)^- = c\log\frac{m_1{}^+}{m_1{}^-}\,\frac{\lambda_1\cdot\dot{\lambda}_1}{|\lambda_1|}$$

Fig. 4 Finite thrust.



The value of the burning time $t - t_0$ may be obtained from the finite, constant mass flow:

$$t - t_0 = (c/k)(m_1{}^+ - m_1{}^-) \tag{46}$$

The solution for the initial value of σ is given by

$$\sigma(t_0) = \frac{m_2{}^+}{m_1{}^-} + \frac{c^2}{k\,m_1{}^-}(m_1{}^+ - m_1{}^-)\frac{\lambda_1\cdot\dot{\lambda}_1}{|\lambda_1|}\log\frac{m_1{}^+}{m_1{}^-} \tag{47}$$

To obtain the initial value of $\dot{\lambda}$, it is necessary to obtain the variational state transition matrix. During coast, we have

$$\ddot{R} = -\mu(R/r^3) \tag{48}$$

The variational equation may be written as

$$\frac{d^2}{dt^2}\frac{\partial R}{\partial \alpha} = -\frac{\mu(\partial R/\partial\alpha)}{r^3} + \frac{3\mu R\cdot(\partial R/\partial\alpha)}{r^5}R \tag{49}$$

Let the α_i be the initial values of R and $\dot{R}$. The solution of Eqs. (48) and (49) is the so-called variational state transition matrix, $\Phi(R, \dot{R})$. The differential equation for the adjoint variable λ is given by

$$\ddot{\lambda} = -\mu\frac{\lambda}{r^3} + 3\mu\frac{\lambda\cdot R}{r^5}R \tag{50}$$

This equation is identical to Eq. (49). Since the initial value of Φ is the unit matrix, it follows that

$$\begin{Bmatrix}\lambda(t)\\ \dot{\lambda}(t)\end{Bmatrix} = \Phi(R, \dot{R})\begin{Bmatrix}\lambda(t_0)\\ \dot{\lambda}(t_0)\end{Bmatrix} \tag{51}$$

The first three equations of Eq. (51) may be evaluated at the terminal time immediately preceding terminal thrust:

$$\lambda_2 = (\partial R/\partial x_0)\lambda_1 + (\partial R/\partial \dot{x}_0)\dot{\lambda}_1 \tag{51a}$$

Solving for $\dot{\lambda}_1$,

$$\dot{\lambda}_1 = \left(\frac{\partial R}{\partial \dot{x}_0}\right)^{-1}\lambda_2 - \left(\frac{\partial R}{\partial \dot{x}_0}\right)^{-1}\left(\frac{\partial R}{\partial \dot{x}_0}\right)\lambda_1 \tag{52}$$

The vector λ_2 may be obtained in a manner similar to λ_1 from the impulsive solution:

$$\lambda_2 = \frac{m_2{}^+}{c}\frac{\dot{R}_2{}^+ - \dot{R}_2{}^-}{\delta v_2} \tag{53}$$

Equation (52) is the required solution for the initial value of $\dot{\lambda}$.

The initial values of λ_1, $\dot{\lambda}_1$, and σ_1 should afford a good approximation for the iterative indirect method.

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